An Algorithmic Approach to Determining Spectra of Orders of (k, g)-Graphs

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Abstract

A (k, g)-*graph* Γ, $k \geq 2, g \geq 3$, is k-regular of girth g. We refer to the complete set of possible orders of connected (k, g)-graphs for each pair of parameters (k, g) as the *spectrum of orders of* (k, g)*-graphs*; or the (k, g)*-spectrum*. The smallest member of the (k, g)-spectrum is the order $n(k, g)$ of a smallest (k, g)-graph; called a (k, g)*-cage*. Determining the complete spectrum of orders of connected (k, g)-graphs for a specific pair (k, g) is extremely difficult as it requires (among other things) determining the cage order $n(k, g)$, which is a notoriously hard problem. This paper provides an algorithmic approach for producing (k, g)-graphs of larger orders from smaller (k, g)-graphs in a manner allowing for repeated applications. We use this approach to determine/estimate the smallest member $N(k, g)$ of a (k, g)-spectrum having the property that starting from $N(k, g)$ all (even; in case of odd k) $n \geq N(k, g)$ belong to the (k, g)-spectrum; i.e., for all (even) $n \geq N(k, g)$, there exists a (k, g)-graph of order n.

Keywords

girth, (k, g)-graphs, (k, g)-spectrum, order of connected (k, g)-graphs

1. Introduction

The *girth* of a (finite) graph Γ is the length of a smallest cycle in Γ; we will always assume that our graphs contain at least some cycles and are connected; they are not trees or forests. A (k, g)-*graph* Γ, $k \geq 2, g \geq 3$, is a connected k-regular of girth g. A (k, g)-graph Γ of minimum order $n(k, g)$ is called a (k, g)-*cage*, and the problem of finding the (k, g)-cages and establishing their orders is called the *Cage Problem*. It is well known that for any given pair of parameters (k, g), there are infinitely many connected (k, g)-graphs [1], which results in an infinite set of orders of (k, g)-graphs for each pair (k, g). More precisely, the results obtained in [1] not only establish the existence of a (k, g)-graph of order $\leq 4\sum_{t=1}^{g-2}(k-1)^t$, for every $k \geq 2, g \geq 3$, but assert the fact that (k, g)-graphs exist for all larger orders as well (for all even larger orders in case of odd k). Unfortunately, the arguments used in [1] are probabilistic and non-constructive.

The complete set of orders of connected (k, g)-graphs for a pair of parameters (k, g) will be referred to as the *spectrum of orders of* (k, g)*-graphs*; or the (k, g)*-spectrum*. Clearly, the smallest element in the (k, g)-spectrum for a specific pair k, g is the order $n(k, g)$ of the (k, g)-cage, and any member of the (k, g)-spectrum is the order of some connected (k, g)-graph. Furthermore, in view of the results in [1], each (k, g)-spectrum contains *a smallest positive integer* $N(k, g)$ *such that all (even, in case of odd* k*) integers* $n \geq N(k, g)$ *are contained in the spectrum.* It is important to note that the number $N(k, g)$ is not necessarily equal to the order $n(k, g)$ of a cage. Specifically, it is well-known that while $n(3, 8) = 30$, there exists no $(3, 8)$-graph of order 32, and thus $N(3, 8) \geq 34$ [3].

The problem of determining the entire (k, g)-spectra for specific parameter pairs (k, g) was probably for the first time earnestly approached in [3]. Since the problem of determining the cage orders $n(k, g)$ is already famously difficult (and solved only for limited sets of parameter pairs), it is clear that one cannot reasonably expect to establish the (k, g)-spectra for all pairs (k, g). For example, the results obtained in [3] include the determination of the (k, g)-spectra for the pairs $(2, g)$ for $g \geq 3$, and $(k, 3)$ and $(k, 4)$ for $k \geq 3$.

In determining the complete spectra of orders of (k, g)-graphs, one usually starts from a Moore graph or a cage and proceeds to construct larger (k, g)-graphs using various construction methods. These methods include excision/addition methods (removal/addition of vertices or edges), voltage graph construction, and others; combined with extensive computer searches. The methods used in [3] include a recursive construction approach based on adding vertices generalizing one of the constructions used in [4] for constructing two $(3, 5)$-graphs of order 12 from the Petersen graph (shown in Figure 1 and 2).

Since the method described in [4] used for constructing two $(3, 5)$-graphs of order 12 from the Petersen graph

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is also the basis of most of the methods presented in here, we include brief descriptions of the constructions. It is interesting to point out that the two constructed $(3, 5)$-graphs of order 12 are the only $(3, 5)$-graphs of this order.

The first $(3, 5)$-graph of order 12 (seen in Figure 1) was constructed from the Petersen graph by selecting a 6-cycle (depicted by vertices $u_1, u_2, u_3, u_4, u_5, u_6$), deleting the three red edges, and adding two vertices of degree 3 to the 6 vertices of the original 6-cycle. By attaching each of the two vertices to one of the end-points of each of the remaining 3 edges of the original 6-cycle, a $(3, 5)$-graph of order 12, which contains no cycle of length less than 5 is obtained (see Figure 1).

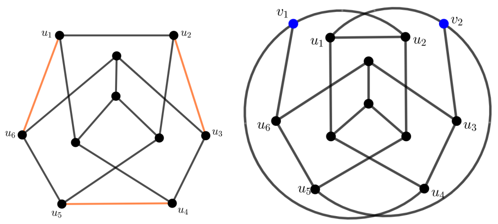

Figure 1: $(3, 5)$-Graph of order 12 (right) obtained from the Petersen graph (left)

The second $(3, 5)$-graph of order 12 (seen in Figure 2) was again constructed from the Petersen graph by selecting a 6-cycle (depicted in red), choosing any two opposing edges of the 6-cycle, subdividing them via introducing a new vertex to each of the selected edges and thereafter joining the two new vertices via an edge.

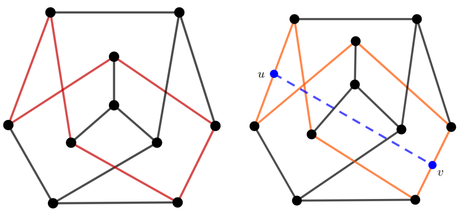

Figure 2: $(3, 5)$-Graph of order 12 (right) obtained from the Petersen graph (left)

Since determining the entire spectrum of orders of (k, g)-graphs for a specific pair (k, g) would also mean determining the cage order $n(k, g)$, instead of attempting to determine the entire spectra, we focus on finding (upper bounds on) the numbers $N(k, g)$ for specific parameter pairs (k, g). Our approach is algorithmic in the sense that we try to design methods that produce larger (k, g)-graphs from smaller (k, g)-graphs in a manner allowing for repeated application.

The paper is organized as follows: Section 2 contains some basic additive properties of (k, g)-spectra, Section 3 introduces constructions based on adding vertices using cycles, and in Section 4 we summarize the usage of the results and constructions of Sections 2 and 3 in an algorithmic way.

2. Additive Properties of (k, g)-Spectra

In this section, we present recursive techniques for building series of (k, g)-graphs from a given starting (k, g)-graph that will provide us with a better understanding of the properties of the (k, g)-spectra.

We begin by introducing a simple construction that will allow us to combine existing (k, g)-graphs into a larger (k, g)-graph.

Lemma 1. *Assume that $k \geq 3$, $g \geq 3$, and let Γ_1, Γ_2 be two (k, g)-graphs of orders m and n, respectively, with the additional property that at least one of the two graphs contains an edge not contained in a g-cycle or contains two distinct g-cycles that differ in at least one edge. Then there exists a (k, g)-graph Γ of order $m + n$.*

Proof. Without loss of generality, we may assume that Γ_1 contains an edge not contained in a g-cycle or or contains two distinct g-cycles that differ in at least one edge. In the first case, let uv be an edge of Γ_1 that is not contained in a g-cycle, and in the second case, let uv be any edge of Γ_1 contained in at most one of the two distinct g-cycles. Further, let $u'v'$ be any edge of Γ_2. Construct Γ to be the graph with vertex set $V(\Gamma_1) \cup V(\Gamma_2)$ and edge set $V(\Gamma_1) \cup V(\Gamma_2)$ minus the selected edges uv, $u'v'$, which are replaced by the edges uv' and $u'v$. Clearly, Γ is a k-regular graph, and to prove the theorem, we just need to prove that the girth of Γ is g. Because of our choice of the edge uv, it is easy to see that Γ contains at least one g-cycle (originally contained in Γ_1). Thus, to finish the proof, we need to show that Γ does not contain cycles shorter than g. Suppose, by means of contradiction, that Γ does contain a cycle $\mathcal{C}$ of length smaller than g. Since both graphs Γ_1 and Γ_2 are of girth g, any such cycle must contain both of the added edges $uv', u'v$, and must be of the form $u, v', u_1^2, u_2^2, \ldots, u_r^2 = u', v, u_1^1, u_2^1, \ldots, u_s^1 = u$, with any two consecutive vertices adjacent, and the vertices with superscript i contained in Γ_i. To obtain the desired contradiction, note that the vertices $u, v, u_1^1, u_2^1, \ldots, u_s^1 = u$ necessarily form a cycle in Γ_1; which is of girth g. Hence, $s + 1 \geq g$, and by a symmetric argument, $r + 1 \geq g$ as well. It follows that the length of $\mathcal{C}$ is at least $2g + 2$ which clearly contradicts the assumption that $\mathcal{C}$ is of length smaller than g. □

Corollary 2. *Let $k \geq 3$ and $g \geq 3$. The (k, g)-spectrum contains all positive integral multiples of $n(k, g)$.*

Proof. Since $k \geq 3$ and $g \geq 3$, it is easy to see that $n(k, g) > g + 1$, and therefore any (k, g)-graph must either contain an edge that does not belong to any g-cycle or contains two distinct g-cycles. Let Γ_1 be a (k, g)-cage of order $n(k, g)$. Since the order of Γ_1 is $n(k, g)$, Γ_1 either contains an edge that does not belong to any g-cycle or it contains two distinct cycles. Thus, based on Lemma 1, it can be used to construct a sequence $\Gamma_i, i \in \mathbb{N}$, defined recursively via constructing Γ_{i+1} by combining Γ_1 and Γ_i, for all $i \geq 2$. Due to Lemma 1, all the graphs Γ_i are (k, g)-graphs of orders $i \cdot n(k, g)$, $i \in \mathbb{N}$. □

Corollary 3. *Let $k \geq 3$ and $g \geq 3$.*

If k is even and there exists an N such that all the consecutive integers $N, N + 1, N + 2, \ldots, N + n(k, g) - 1$ belong to the (k, g)-spectrum, then all $n \geq N$ belong to this spectrum, and $N \geq N(k, g)$.

If k is odd and there exists an even N such that all the consecutive even integers $N, N + 2, N + 4, \ldots, N + n(k, g) - 2$ belong to the (k, g)-spectrum, then all even $n \geq N$ belong to this spectrum, and $N \geq N(k, g)$.

Proof. In case of even k, all the graphs Γ_i constructed in the proof of Corollary 2 either contain an edge that does not belong to any g-cycle or contain two distinct g-cycles. Since $N, N + 1, N + 2, \ldots, N + n(k, g) - 1$ belong to the (k, g)-spectrum, there exist (k, g)-graphs Δ_j, $N \leq j \leq N + n(k, g) - 1$, of orders $N, N + 1, N + 2, \ldots, N + n(k, g) - 1$, respectively. Using Lemma 1 to combine the graphs Δ_j with the graphs Γ_i yields the desired family of (k, g)-graphs whose orders cover all positive integers greater than or equal to N.

If k is odd, the order of any (k, g)-graph is even, while the rest of the proof follows essentially along the same line as the case of even k. □

3. Adding Vertices Using Cycles

In this section, we introduce recursive constructions of (k, g)-graphs assuming the existence of an additional cycle of certain length greater than g in the starting graph. We first illustrate our approach by (re)determining the order spectrum of the $(3, 5)$-graphs (already achieved in [3]).

Lemma 4. *Let Γ be a $(3, 5)$-graph of order n containing a 6-cycle. Then there exists a $(3, 5)$-graph of order $n + 2$ that also contains a 6-cycle.*

Proof. Let $\mathcal{C}$ be a 6-cycle in Γ consisting of the vertices $u_1, u_2, u_3, u_4, u_5, u_6$, with any two consecutive vertices adjacent. To construct the $(3, 5)$ graph Γ^* from Γ, remove the three edges u_2u_3, u_4u_5, and u_6u_1, and add two new vertices v_1, v_2 together with six new edges $u_2v_1, u_4v_1, u_6v_1, u_1v_2, u_3v_2$ and u_5v_2. Clearly, Γ^* is 3-regular, and contains a 6-cycle, e.g., $v_1, u_6, u_5, v_2, u_3, u_4, v_1$. It remains to prove that Γ^* does not contain cycles shorter than 5. To see that, note first that since the girth of Γ is 5, and the vertices u_2, u_3 are adjacent in Γ, after removing their shared edge, they must be of distance at least 4 (as otherwise we would have a 3- or a 4-cycle in Γ). As no path between these two vertices of length shorter than 3 has been added into Γ^*, it is easy to see that the distance between u_2 and u_3 in Γ^* is still at least 3. The same is true for the pairs u_4, u_5 and u_6, u_1, and even more importantly, the length of any path between any two neighbors of v_1 not using v_1 is at least 3, as their distance in Γ was originally 2. The same holds for any two neighbors of v_2. By means of contradiction, assume now that Γ^* contains a cycle of length shorter than 5. Any such cycle must contain at least one of the new vertices v_1, v_2. Should such cycle contain both vertices v_1 and v_2, since they are of distance 3, it would have to be at least a 6-cycle. If it contained just one of them, it would also have to contain its neighbors, but any two neighbors of any of the vertices v_1, v_2 are of distance at least 3 and hence the length of such cycle would have to be at least 5. Hence, Γ^* contains no cycles shorter than 5 and the proof is complete. □

Corollary 5. *The spectrum of orders of $(3, 5)$-graphs is the set of all even integers greater than or equal to 10.*

Proof. Start with the Petersen graph. It contains a 6-cycle (see, for example, the outer 6-cycle in Figure 1). Add the vertices v_1, v_2 and the six new edges as described in Lemma 4. Choose one of the two new 6-cycles, and note that it only contains two of the three original edges of the 6-cycle in the Petersen, and so no further change of the selected 6-cycle will affect the fact that the third edge originally contained in the 6-cycle of the Petersen graph is (and will remain) contained in a 5-cycle. By always adding new vertices to 6-cycles added in the previous step, we will construct an infinite sequence of $(3, 5)$-graphs (all containing at least one 5-cycle and at least one 6-cycle) of orders $10, 12, 14, 16, \ldots$. □

Interestingly, even though the above construction cannot be used in case of $(3, 6)$-graphs as it may occasionally create a 5-cycle, there are cases where applying the construction to a specific 6-cycle in a $(3, 6)$-graph of order n yields a $(3, 6)$-graph of order $n + 2$. One such case appears when one attaches two vertices to a 6-cycle in the Heawood graph. See Figure 3.

Although the construction described in Lemma 4 can be generalized into infinitely many new (similar) lemmas, we only introduce two such generalizations. The reason for not trying to present all such lemmas is the fact that in general we do not have such a convenient starting point as is the Petersen graph in Corollary 5, and hence

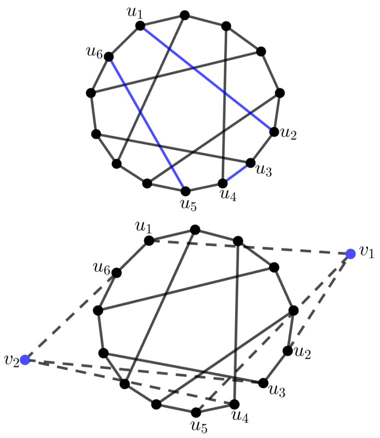

Figure 3: Example of $(3, 6)$-Graph of order 16 (right) obtained from the Heawood graph (left)

we cannot use the forthcoming lemmas to obtain results of similar strength to those of Corollary 5. Even though these lemmas do allow one to add two vertices to an existing $(3, g)$- or $(4, g)$-graph in a way resulting in a larger $(3, g)$- or $(4, g)$-graph, finding graphs satisfying the required properties is hard, and one does not have any guarantee that the process of adding two vertices can be recursively repeated. Therefore, these lemmas are best seen as starting points for computer assisted constructions.

Lemma 6. *Let $g \geq 8$ be even, and let Γ be a $(3, g)$-graph of order n containing a cycle $\mathcal{C}$ of length at least $(\frac{3g}{2} - 3)$ which contains edges u_1u_2, u_3u_4, and u_5u_6 such that the distances between the pair of vertices u_1, u_6, the pair of vertices u_2, u_3 and the pair of vertices u_4, u_5 in Γ are at least $\frac{g}{2} - 2$ and the distances between any two of the vertices u_1, u_3, u_5 and the distances between any two of the vertices u_2, u_4, u_6 in the graph $\Gamma - u_1u_2 - u_3u_4 - u_5u_6$ (Γ with three edges removed) are at least $g - 2$. In addition, let Γ contain at least one g-cycle that does not contain any of the edges u_1u_2, u_3u_4, or u_5u_6. Then there exists a $(3, g)$-graph Γ^* of order $n + 2$.*

Proof. As stated already, this lemma is an analogue of Lemma 4. Let $\mathcal{C}$ be a cycle in Γ of length at least $(\frac{3g}{2} - 3)$, and let u_1u_2, u_3u_4 and u_5u_6 be edges of $\mathcal{C}$ with the properties described in the lemma. The desired graph Γ^* can then be constructed from Γ by removing the edges u_1u_2, u_3u_4 and u_5u_6, and adding new vertices v_1 and v_2 with v_1 adjacent to the vertices u_1, u_3 and u_5 and v_2 adjacent to the vertices u_2, u_4 and u_6 (in the same manner as in the proof of Lemma 4). We leave it to the reader to verify that since Γ is of girth g, Γ^* contains no cycles shorter than g and contains at least one g-cycle (the g-cycle not containing the three removed edges). □

We feel obliged to note here that the existence of the desired $(\frac{3g}{2} - 3)$ cycle in a $(3, g)$-cage is not guaranteed. For example, if the Tutte-Coxeter graph, the $(3, 8)$-cage of order 30, contained cycles with the properties described in Lemma 6, it would imply the existence of a $(3, 8)$-graph of order 32. However, as already pointed out in the Introduction, it is well known that no such graph exists.

In particular, since the Tutte-Coxeter graph is the point-line incidence graph of a generalized quadrangle of order 2, it is bipartite, and therefore contains no 9-cycles, while the minimal length of the cycle $\mathcal{C}$ required in the lemma is at least $(\frac{3 \cdot 8}{2} - 3) = 9$. On the other hand, the existence of a $(3, 8)$-graph of order 34 containing a 9-cycle of the desired properties would yield a $(3, 8)$-graph of order 36 and might be the beginning of the continuous part of the $(3, 8)$-spectrum (i.e., it might be the case that $N(3, 8) = 34$). At this point, we have not yet determined whether such graph on 34 vertices exists.

We conclude the section with one more analogue of Lemma 4.

Lemma 7. *Let $g \geq 8$ be even, and let Γ be a $(4, g)$-graph of order n containing a cycle $\mathcal{C}$ of length at least $(2g - 2)$ which contains edges u_1u_2, u_3u_4, u_5u_6, and u_7u_8 such that the distances between the pair of vertices u_1, u_8, the pair of vertices u_2, u_3, the pair of vertices u_4, u_5 and the pair of vertices u_6, u_7 in Γ are at least $\frac{g}{2} - 2$ and the distances between any two of the vertices u_1, u_3, u_5, u_7 and the distances between any two of the vertices u_2, u_4, u_6, u_8 in the graph $\Gamma - u_1u_2 - u_3u_4 - u_5u_6 - u_7u_8$ (Γ with four edges removed) are at least $g - 2$. In addition, let Γ contain at least one g-cycle that does not contain any of the edges u_1u_2, u_3u_4, u_5u_6, or u_7u_8. Then there exists a $(4, g)$-graph Γ^* of order $n + 2$.*

Proof. We leave it to the reader to verify that the graph Γ^* constructed from Γ described in the lemma via the removal of the edges u_1u_2, u_3u_4, u_5u_6, u_7u_8, and adding vertices v_1, v_2 adjacent to u_1, u_3, u_5, u_7 and u_2, u_4, u_6, u_8 respectively, is a $(4, g)$-graph. □

4. Algorithmic Approach to Determining the (k, g)-Spectra

Let us begin the section by combining the approaches introduced in Sections 2 and 3. Recall that for any pair of positive integers m and n, the $\gcd(m, n)$ is a sum of integral multiples of m and n, and moreover, there exists an integer N such that all integral multiples of $\gcd(m, n)$ greater than or equal to $N \cdot \gcd(m, n)$ are sums of positive integral multiples of m and n. Hence, in case of even k and any $g \geq 3$, the existence of (k, g)-graphs of relatively prime orders m and n yields, via using Lemma 1, the existence of $N(k, g)$ as defined in the Introduction (and proved to exist in [1]). Similarly, in case of odd k and any $g \geq 3$, the existence of (k, g)-graphs of orders $2m$ and $2n$, with m and n relatively prime, yields by Lemma 1 the existence of $N(k, g)$ again.

Since $\gcd(n, n+1) = 1$ and $\gcd(2n, 2n+2) = 2$, Section 3 provides one with the possibility of constructing consecutive pairs of even orders yielding the existence of $N(k, g)$. The only drawback of using the lemmas contained in Section 3 is the requirement of using graphs containing the specific cycles of length greater than g. Since no universal constructions of such graphs are known, one has to rely on computer searches.

In this section, we present an algorithm for searching for graphs described in the previous sections. We implemented our algorithm (Algorithm 1) in the system for discrete computational algebra - *GAP* version *4.12.2* and used two packages: *DIGRAPHS* and *GRAPE*.

The input for our algorithm is the `graph` object from *GRAPE* package. In a for-cycle, we go through all combinations of three edges, which we will later remove. We store the edges in a `removedEdges` variable. The names of vertices of those edges are saved in variable `verticesToReconnect` as we will need their names to connect them with the new two vertices. In the next step we remove the edges in `removedEdges` with the function `RemoveEdges()` and obtain a new graph object (`graphWithoutEdges`). Then, we add the two new vertices with `AddVertices()`, which returns a graph object (`graphWithVertices`). To obtain all possible cubic graphs in this iteration through combinations of edges, we connect the two new vertices with vertices in `verticesToReconnect` with the function `ConnectVertices()`. The functions `RemoveEdges()`, `AddVertices()` and `ConnectVertices()` use commands from the *DIGRAPHS* package. We store all graphs with the same girth as the original graph in variable `graphsWithCorrectGirth`. If the list of graphs with required girth is not empty, we check for isomorphic graphs and keep only non isomorphic ones. The function returns a list of non isomorphic graphs with the excess 2.

Using Algorithm 1, starting from the Petersen graph, we found 2 graphs with two additional vertices, 8 graphs with four additional vertices, and 48 graphs with six additional vertices. Starting from the McGee graph, we found 1 graph with one additional vertex, 1 graph with four additional vertices, and 6 graphs with six additional vertices. For the $(3, 11)$-cage, we found 2 graphs with two additional vertices.

Algorithm 1 Remove3EdgesAdd2Vertices(graph)

```
for all combinations of 3 edges as removedEdges do
  verticesToReconnect = Flat(removedEdges)
  graphWithoutEdges = RemoveEdges(graph, removedEdges)
  graphWithVertices = AddVertices(graphWithoutEdges, 2)
  possibleGraphs = ConnectVertices(graphWithVertices, verticesToReconnect)
  graphsWithCorrectGirth = CheckGirth(possibleGraphs, Girth(graph))
  if graphsWithCorrectGirth not empty then
    newGraphs = GetUnisomorphicGraphs(graphsWithCorrectGirth)
  end if
end for
return newGraphs
```

5. Conclusion

In conclusion, let us point out that the above computer implementation of our algorithms has been developed in parallel and somewhat independently of our theoretical results. That did not leave enough time for us to take full advantage of these programs. Thus the above summary of results obtained using our programs should be seen as just examples of the use of our techniques. For example, as stated in the previous section, in order to obtain a characterization of the order spectrum of $(3, 8)$-graphs, we would need to find a smallest $(3, 8)$-graph containing a 9-cycle (which must be of order at least 34). As this does not seem to be out of reach, a more persistent use of these programs might eventually lead to new results.

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